Special Summer Integrated Learning Programme

Mathematics

Fundamentals

Week 1 Day 1 PM

SINGAPORE UNIVERSITY OF TECHNOLOGY AND DESIGN

Outline

1. Inequalities
2. The Coordinate Plane, Graphs of Equations, Circles
3. Lines
4. Summary
5. Equations [Class Participation Assignment]

Inequalities

Some problems in algebra lead to inequalities instead of equations. To solve an inequality that contains a variable means to find all values of the variable that make the inequality true.

Unlike an equation, an inequality generally has infinitely many solutions. For instance, consider the inequality

$$4x + 7 \leq 19$$

x	$4x + 7 \leq 19$
1	$11 \leq 19$ ✓
2	$15 \leq 19$ ✓
3	$19 \leq 19$ ✓
4	$23 \leq 19$ ✗
5	$27 \leq 19$ ✗

From the table, we see that some values of x satisfy the inequality and some values don't.

Rules for inequalities

We use the following rules to isolate the variable on one side of the inequality sign. Here we state the rules for inequalities involving the symbol $\leq$, but they apply to all four inequality symbols $\leq, <, \geq, >$.

RULES FOR INEQUALITIES

Rule	Description
1. $A \leq B \quad \Leftrightarrow \quad A + C \leq B + C$	**Adding** the same quantity to each side of an inequality gives an equivalent inequality.
2. $A \leq B \quad \Leftrightarrow \quad A - C \leq B - C$	**Subtracting** the same quantity from each side of an inequality gives an equivalent inequality.
3. If $C > 0$, then $A \leq B \quad \Leftrightarrow \quad CA \leq CB$	**Multiplying** each side of an inequality by the same *positive* quantity gives an equivalent inequality.
4. If $C < 0$, then $A \leq B \quad \Leftrightarrow \quad CA \geq CB$	**Multiplying** each side of an inequality by the same *negative* quantity *reverses the direction* of the inequality.
5. If $A > 0$ and $B > 0$, then $A \leq B \quad \Leftrightarrow \quad \frac{1}{A} \geq \frac{1}{B}$	**Taking reciprocals** of each side of an inequality involving *positive* quantities *reverses the direction* of the inequality.
6. If $A \leq B$ and $C \leq D$, then $A + C \leq B + D$	Inequalities can be added.
7. If $A \leq B$ and $B \leq C$, then $A \leq C$	Inequality is transitive.

Rules for inequalities

Note that the direction of the inequality is reversed in rules no. 4 and 5. In reality, these are actually due to swapping sides:

4.

$$\begin{aligned} A \leq B \quad &\Leftrightarrow \quad A + (-A - B) \leq B + (-A - B) \\ \Leftrightarrow \quad -B \leq -A \quad &\Leftrightarrow \quad -A \geq -B. \end{aligned}$$

5. For $A, B > 0$

$$\begin{aligned} A \leq B \quad &\Leftrightarrow \quad A \cdot \frac{1}{AB} \leq B \cdot \frac{1}{AB} \\ \Leftrightarrow \quad \frac{1}{B} \leq \frac{1}{A} \quad &\Leftrightarrow \quad \frac{1}{A} \geq \frac{1}{B}. \end{aligned}$$

Solving linear inequalities

An inequality is linear if both sides of the inequality are polynomials of degree 1. To solve a linear inequality, we isolate the variable on one side of the inequality sign.

E.g.: To solve $4x - 7 < 8 + 9x$, we shift all the x terms to the right-hand side:

$$\begin{aligned} & 4x - 7 < 8 + 9x & \Leftrightarrow \quad & -7 - 8 < 9x - 4x \\ \Leftrightarrow \quad & -15 < 5x & \Leftrightarrow \quad & -3 < x. \end{aligned}$$

Hence, the solutions are all real numbers $x > -3$, i.e. the solution set is the interval $(-3, \infty)$.

Fun fact: Linear programming is a branch of mathematics that studies linear inequalities that involve multiple variables. Industries use linear programming to model transportation, energy, telecommunications, and manufacturing. It has proven useful in modeling diverse types of problems in planning, routing, scheduling, assignment, and design.

Solving nonlinear inequalities

To solve inequalities involving squares and other powers of the variable, we use the following guidelines

1. **Move all terms to one side**.
2. **Factor** the nonzero side of the inequality. You need to factorize both the numerator and the denominator if you have a fractional expression.
3. **Find the intervals** of which each factor is positive/negative.
4. **Make a table**, where the columns are intervals with endpoints from the previous step and the rows are the corresponding signs of the factors in those intervals.
5. **Solve** the inequality by multiplying all the signs along a column. If a factor is at the denominator, put down "undefined" in those columns where this factor is equal to zero (because we cannot divide by 0).

We'll illustrate the guidelines using examples in the next two slides.

Solving nonlinear inequalities - example 1

E.g.: To solve $x^2 - 5x \leq -6$, we first **move** all terms to the left-hand side and then **factorize** it:

$$x^2 - 5x \leq -6 \quad \Leftrightarrow \quad x^2 - 5x + 6 \leq 0 \quad \Leftrightarrow \quad (x-3)(x-2) \leq 0.$$

Next, $x - 3 > 0$ over the **interval** $(3, \infty)$ and $x - 2 > 0$ over the **interval** $(2, \infty)$. We then **make the following table**:

	$x < 2$	$x = 2$	$2 < x < 3$	$x = 3$	$x > 3$
$x - 3$	$-$	$-$	$-$	0	$+$
$x - 2$	$-$	0	$+$	$+$	$+$
$(x-3)(x-2)$	$+$	**0**	**–**	**0**	$+$

Since we want to solve $(x-3)(x-2) \leq 0$, the solutions correspond to the columns that give $-$ or 0 as the product of all the factors, i.e. it's the interval $2 \leq x \leq 3$.

Solving nonlinear inequalities - example 2

E.g.: To solve $\frac{(x-3)(x-2)}{x+1} \geq 0$, we note that $x-3>0$ over $(3,\infty)$, $x-2>0$ over $(2,\infty)$, and $x+1>0$ over $(-1,\infty)$. We then make the following table:

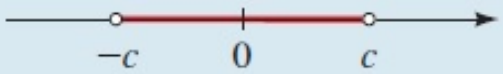
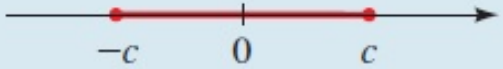
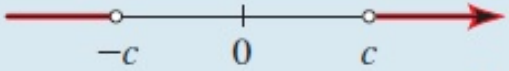
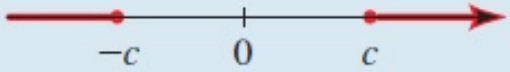
	$x<-1$	$x=-1$	$-1<x<2$	$x=2$	$2<x<3$	$x=3$	$x>3$
$x-3$	$-$	$-$	$-$	$-$	$-$	0	$+$
$x-2$	$-$	$-$	$-$	0	$+$	$+$	$+$
$x+1$	$-$	0	$+$	$+$	$+$	$+$	$+$
$\frac{(x-3)(x-2)}{x+1}$	$-$	undefined	$+$	0	$-$	0	$+$

Since we want to solve $\frac{(x-3)(x-2)}{x+1} \geq 0$, the solutions correspond to the columns that give 0 or $+$ as the product of all the factors, i.e. it's the set $(-1,2] \cup [3,\infty)$.

Absolute value inequalities

To solve inequalities that involve absolute values, apply the following properties to remove the absolute value sign:

PROPERTIES OF ABSOLUTE VALUE INEQUALITIES

Inequality	Equivalent form	Graph
1. $\lvert x \rvert < c$	$-c < x < c$	
2. $\lvert x \rvert \le c$	$-c \le x \le c$	
3. $\lvert x \rvert > c$	$x < -c$ or $c < x$	
4. $\lvert x \rvert \ge c$	$x \le -c$ or $c \le x$	

⚠ Note that in general $A < B < C$ means $A < B$ and $B < C$, i.e. the solution set for $A < B < C$ is the intersection of the solution sets of $A < B$ and $B < C$. For instance, $-2 < x < 2$ means $-2 < x$ and $x < 2$, hence the solution set is $(-2, \infty) \cap (-\infty, 2) = (-2, 2)$. On the other hand, when solving $A < B$ or $C < D$, we take the union of the solution sets of $A < B$ and $C < D$.

Absolute value inequalities - example

E.g.: To solve $|x-5|>2$, we first apply property 3:

$$\begin{aligned} |x-5|>2 \quad &\Leftrightarrow \quad x-5<-2 \quad \text{or} \quad x-5>2 \\ &\Leftrightarrow \quad x<3 \quad \text{or} \quad x>7. \end{aligned}$$

Thus, the solution set is $(-\infty,3)\cup(7,\infty)$.

E.g.: To solve $|2x-3|\leq x$, we first apply property 2:

$$\begin{aligned} |2x-3|\leq x \quad &\Leftrightarrow \quad -x\leq 2x-3\leq x \\ &\Leftrightarrow \quad -x\leq 2x-3 \quad \text{and} \quad 2x-3\leq x \\ &\Leftrightarrow \quad 3\leq 3x \quad \text{and} \quad x\leq 3 \\ &\Leftrightarrow \quad 1\leq x \quad \text{and} \quad x\leq 3. \end{aligned}$$

Thus, the solution set is $[1,\infty)\cap(-\infty,3]=[1,3]$.

Activity 1

Find the solutions of the following inequalities.

(a) $2(7x-3) \leq 12x+16$

(b) $-1 < 2x-5 < 7$

(c) $x^2+2x > 3$

(d) $x^2(x^2-9) \leq 0$

(e) $\frac{x}{x+1} > 3x$

(f) $\frac{x+2}{x+3} \leq \frac{x-1}{x-2}$

(g) $3-|2x+4| \leq 1+x$

(h) $|3x+5| \leq x-4$

Activity 1: solution

(a)

$$\begin{aligned} 2(7x-3) \leq 12x+16 \quad &\Leftrightarrow \quad 7x-3 \leq \frac{1}{2}(12x+16) \\ \Leftrightarrow \quad 7x-3 \leq 6x+8 \quad &\Leftrightarrow \quad 7x-6x \leq 8+3 \\ &\Leftrightarrow \quad x \leq 11. \end{aligned}$$

(b)

$$\begin{aligned} -1 < 2x-5 < 7 \quad &\Leftrightarrow \quad -1+5 < 2x < 7+5 \\ \Leftrightarrow \quad \frac{4}{2} < x < \frac{12}{2} \quad &\Leftrightarrow \quad 2 < x < 6. \end{aligned}$$

Activity 1: solution

(c)

$$x^2 + 2x > 3 \quad \Leftrightarrow \quad x^2 + 2x - 3 > 0$$
$$\Leftrightarrow \quad (x+3)(x-1) > 0.$$

	$x < -3$	$x = -3$	$-3 < x < 1$	$x = 1$	$x > 1$
$x+3$	$-$	0	$+$	$+$	$+$
$x-1$	$-$	$-$	$-$	0	$+$
$(x+3)(x-1)$	$+$	0	$-$	0	$+$

So the solutions are $x < -3$ or $x > 1$, i.e. the solution set is $(-\infty, -3) \cup (1, \infty)$.

(d) When $x \neq 0$, we have $x^2 > 0$ and so

$$x^2(x^2 - 9) \leq 0 \quad \Leftrightarrow \quad x^2 - 9 \leq 0$$
$$\Leftrightarrow \quad x^2 \leq 9 \quad \Leftrightarrow \quad |x| \leq 3.$$

When $x = 0$, the inequality $x^2(x^2 - 9) \leq 0$ is true. Thus, the solutions are $x = 0$ or $|x| \leq 3$, i.e. $-3 \leq x \leq 3$.

Activity 1: solution

(e)

$$\frac{x}{x+1} > 3x \quad \Leftrightarrow \quad \frac{x}{x+1} - 3x > 0$$

$$\Leftrightarrow \quad \frac{x - 3x(x+1)}{x+1} > 0 \quad \Leftrightarrow \quad \frac{-3x^2 - 2x}{x+1} > 0$$

$$\Leftrightarrow \quad \frac{3x^2 + 2x}{x+1} < 0 \quad \Leftrightarrow \quad \frac{x(3x+2)}{x+1} < 0$$

	$x < -1$	$x = -1$	$-1 < x < -\frac{2}{3}$	$x = -\frac{2}{3}$	$-\frac{2}{3} < x < 0$	$x = 0$	$x > 0$
$x+1$	$-$	0	$+$	$+$	$+$	$+$	$+$
$3x+2$	$-$	$-$	$-$	0	$+$	$+$	$+$
x	$-$	$-$	$-$	$-$	$-$	0	$+$
$\frac{x(3x+2)}{x+1}$	$-$	undefined	$+$	0	$-$	0	$+$

So the solutions are $x < -1$ or $-\frac{2}{3} < x < 0$, i.e. the solution set is $(-\infty, -1) \cup \left(-\frac{2}{3}, 0\right)$.

Activity 1: solution

(f)

$$\frac{x+2}{x+3} \leq \frac{x-1}{x-2} \quad \Leftrightarrow \quad \frac{x+2}{x+3} - \frac{x-1}{x-2} \leq 0$$

$$\Leftrightarrow \quad \frac{(x+2)(x-2)-(x-1)(x+3)}{(x+3)(x-2)} \leq 0 \quad \Leftrightarrow \quad \frac{-2x-1}{(x+3)(x-2)} \leq 0$$

$$\Leftrightarrow \quad \frac{2x+1}{(x+3)(x-2)} \geq 0$$

	$x < -3$	$x = -3$	$-3 < x < -\frac{1}{2}$	$x = -\frac{1}{2}$	$-\frac{1}{2} < x < 2$	$x = 2$	$x > 2$
$x+3$	$-$	0	$+$	$+$	$+$	$+$	$+$
$2x+1$	$-$	$-$	$-$	0	$+$	$+$	$+$
$x-2$	$-$	$-$	$-$	$-$	$-$	0	$+$
$\frac{2x+1}{(x+3)(x-2)}$	$-$	undefined	$+$	0	$-$	undefined	$+$

So the solutions are $-3 < x \leq -\frac{1}{2}$ or $x > 2$, i.e. the solution set is $\left(-3, -\frac{1}{2}\right] \cup (2, \infty)$.

Activity 1: solution

(g)

$$
\begin{aligned}
& 3-|2x+4| \leq 1+x && \Leftrightarrow \quad 3-1-x \leq |2x+4| \\
\Leftrightarrow \quad & |2x+4| \geq 2-x \\
\Leftrightarrow \quad & 2x+4 \leq -(2-x) && \text{or} \quad 2x+4 \geq 2-x \\
\Leftrightarrow \quad & 2x-x \leq -2-4 && \text{or} \quad 2x+x \geq 2-4 \\
\Leftrightarrow \quad & x \leq -6 && \text{or} \quad x \geq -\frac{2}{3}.
\end{aligned}
$$

So the solutions are $x \leq -6$ or $x \geq -\frac{2}{3}$, i.e. the solution set is $(-\infty, -6] \cup \left[-\frac{2}{3}, \infty\right)$.

Activity 1: solution

(h)

$$
\begin{aligned}
& |3x+5| \leq x-4 && \Leftrightarrow && -(x-4) \leq 3x+5 \leq x-4 \\
\Leftrightarrow\ & -(x-4) \leq 3x+5 && \text{and} && 3x+5 \leq x-4 \\
\Leftrightarrow\ & 4-5 \leq 3x+x && \text{and} && 3x-x \leq -4-5 \\
\Leftrightarrow\ & -\frac{1}{4} \leq x && \text{and} && x \leq -\frac{9}{2}.
\end{aligned}
$$

So the solutions are all real numbers x that satisfy $-\frac{1}{4} \leq x$ and $x \leq -\frac{9}{2}$ simultaneously, i.e. the solution set is $\left(-\infty, -\frac{9}{2}\right] \cap \left[-\frac{1}{4}, \infty\right) = \emptyset$. Hence, the inequality has no solutions. In other words, $|3x+5| > x-4$ for all real numbers x,. You can visualize this with Geogebra.

Outline

1. Inequalities
2. The Coordinate Plane, Graphs of Equations, Circles
3. Lines
4. Summary
5. Equations [Class Participation Assignment]

The coordinate plane

Just as points on a line can be identified with real numbers to form the real line, points in a plane can be identified with ordered pairs of numbers to form the ***coordinate plane*** or ***Cartesian plane*** (named after René Descartes).

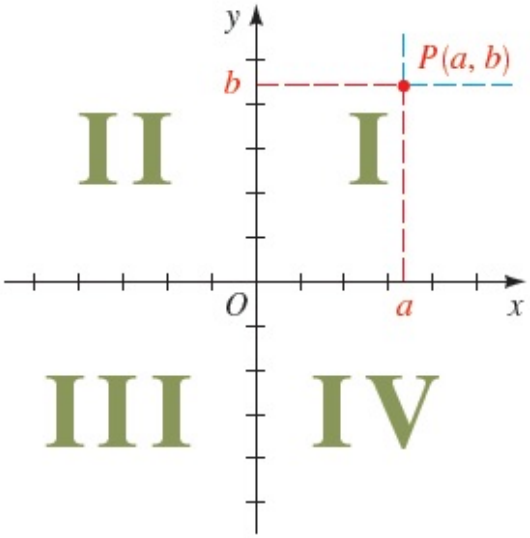


FIGURE 1

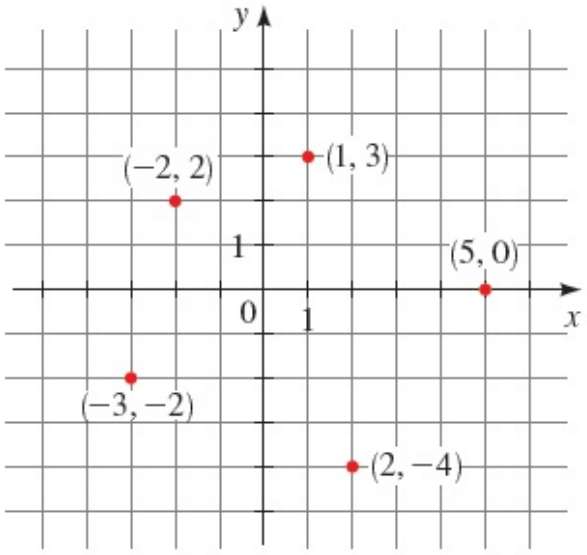


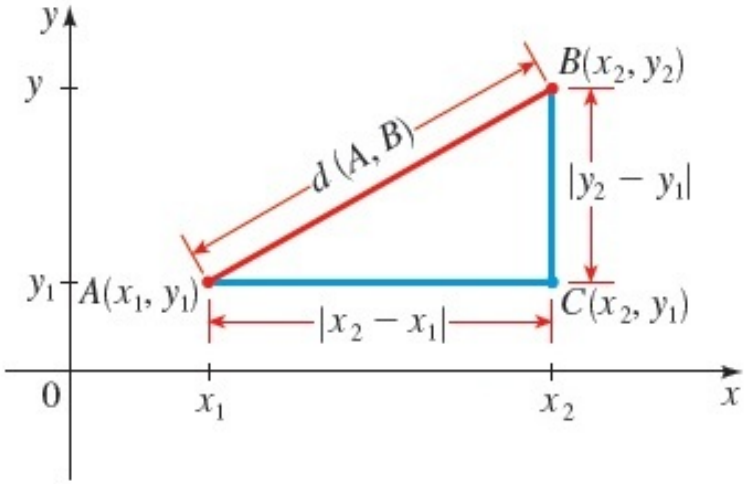
FIGURE 2

Figure 1 shows the 4 quadrants I, II, III, IV, with O as the ***origin***. Any point P in the coordinate plane can be pinpointed by a unique ordered pair of numbers (a, b). a is called the ***x-coordinate*** of P, b is called the ***y-coordinate*** of P. See Figure 2 for some examples.

The distance formula

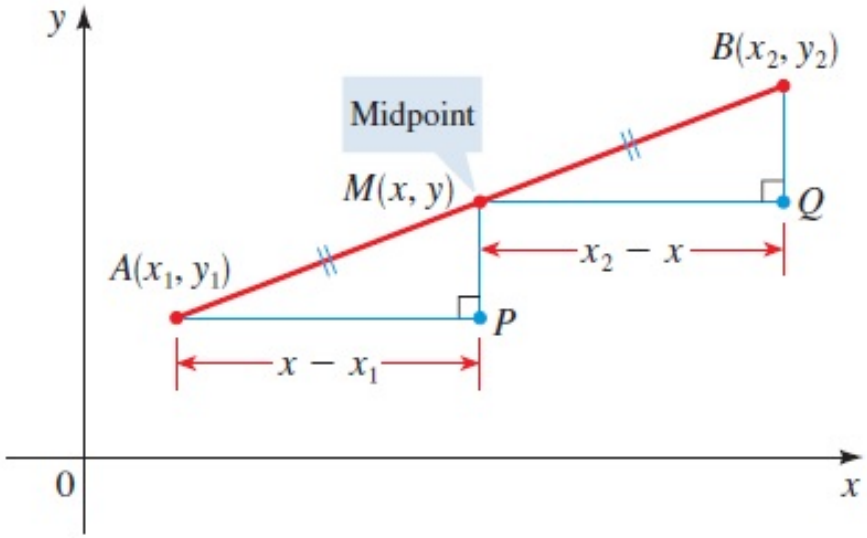
By Pythagoras' theorem, we can compute the distance between the points $A(x_1, y_1)$ and $B(x_2, y_2)$ in the plane with the following formula:

$$d(A, B) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$



The midpoint formula

The ***midpoint*** M of the line segment that joins the point $A(x_1, y_1)$ and $B(x_2, y_2)$ is the point on that line such that $d(A, M) = d(M, B)$, i.e. it's the point that cuts the line segment AB into two equal halves.

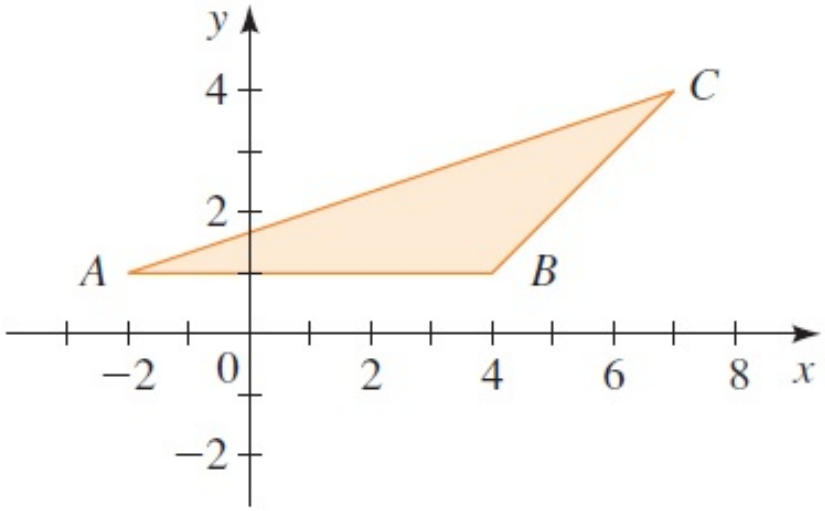


The coordinates (x, y) of the midpoint of the line segment that joins the point $A(x_1, y_1)$ and $B(x_2, y_2)$ is given by

$$(x, y) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right).$$

Activity 2

Find the perimeter and the area of the triangle ABC.



Activity 2: solution

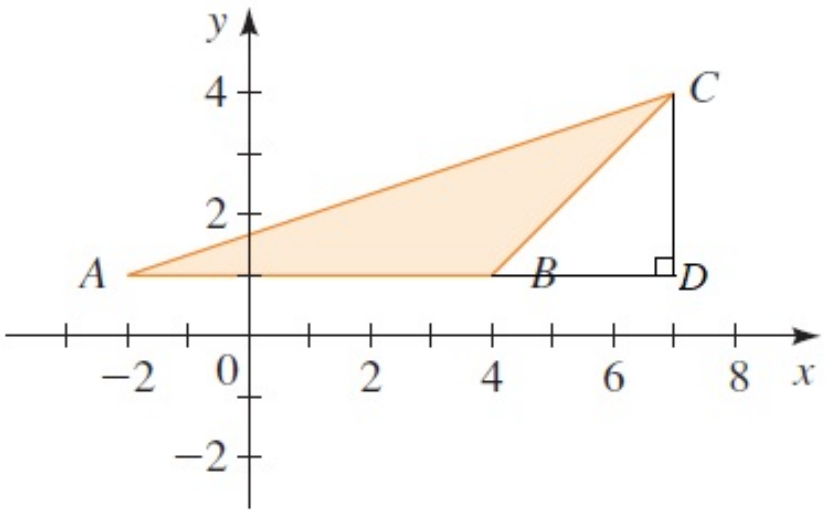
The coordinates are $A(-2,1)$, $B(4,1)$ and $C(7,4)$. So

$$d(A,B) = 4-(-2) = 6,$$
$$d(B,C) = \sqrt{(7-4)^2+(4-1)^2} = \sqrt{9+9} = 3\sqrt{2},$$
$$d(A,C) = \sqrt{(7-(-2))^2+(4-1)^2} = \sqrt{81+9} = 3\sqrt{10}.$$

So the perimeter of ABC is
$d(A,B)+d(B,C)+d(A,C) = 6+3\sqrt{2}+3\sqrt{10}.$

Activity 2: solution

As for the area of ABC, it's equal to $\frac{1}{2}$ times the base $d(A, B)$ times the height $d(D, C)$.



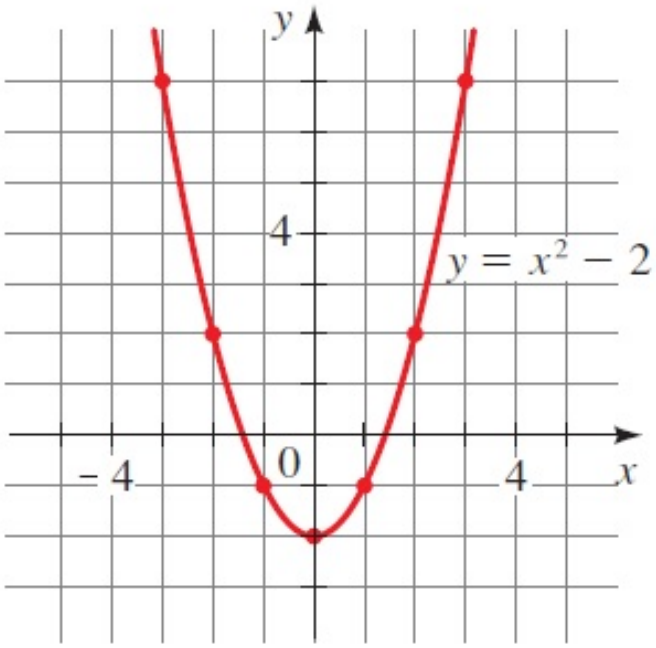
Since $d(A, B) = 4 - (-2) = 6$, $d(D, C) = 4 - 1 = 3$

$$\begin{aligned} \text{area of } ABC &= \frac{1}{2} \times d(A, B) \times d(D, C) \\ &= \frac{1}{2} \times 6 \times 3 = 9. \end{aligned}$$

Graphs of Equations in Two Variables

The ***graph*** of an equation in x and y is the set of all points (x, y) in the coordinate plane that satisfy the equation.

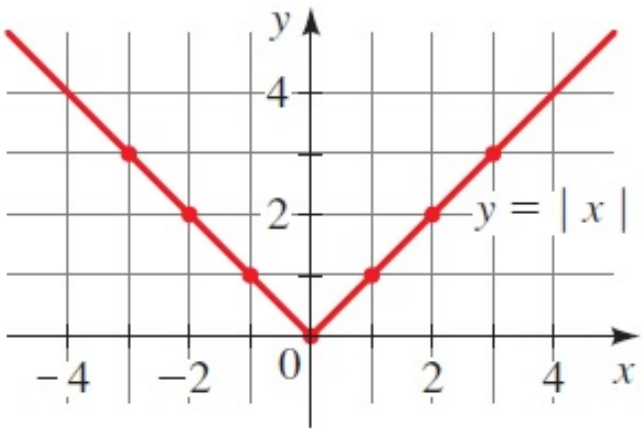
E.g.:



x	$y = x^2 - 2$	(x, y)
-3	7	$(-3, 7)$
-2	2	$(-2, 2)$
-1	-1	$(-1, -1)$
0	-2	$(0, -2)$
1	-1	$(1, -1)$
2	2	$(2, 2)$
3	7	$(3, 7)$

Graphs of Equations in Two Variables

E.g.:



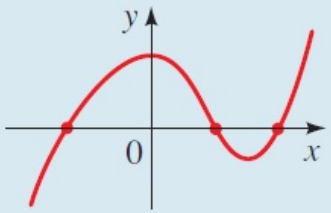
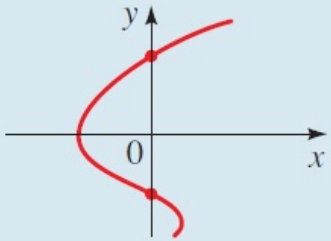
x	$y = \lvert x \rvert$	(x, y)
-3	3	$(-3, 3)$
-2	2	$(-2, 2)$
-1	1	$(-1, 1)$
0	0	$(0, 0)$
1	1	$(1, 1)$
2	2	$(2, 2)$
3	3	$(3, 3)$

You can use Geogebra to plot graphs easily. Try it to plot the graphs of the following equations:

- $y = \frac{1}{1+x^2}$
- $y = x - \sin(x)$
- $y^{\frac{1}{3}} + x^{\frac{1}{3}} = 1$
- $x^3 + y^3 - xy = 0$

Intercepts

DEFINITION OF INTERCEPTS

Intercepts	How to find them	Where they are on the graph
***x*-intercepts:** The x-coordinates of points where the graph of an equation intersects the x-axis	Set $y = 0$ and solve for x	
***y*-intercepts:** The y-coordinates of points where the graph of an equation intersects the y-axis	Set $x = 0$ and solve for y	

Circles

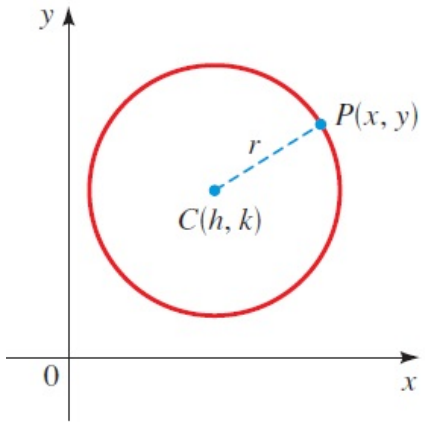
EQUATION OF A CIRCLE

An equation of the circle with center (h, k) and radius r is

$$(x - h)^2 + (y - k)^2 = r^2$$

This is called the **standard form** for the equation of the circle. If the center of the circle is the origin $(0, 0)$, then the equation is

$$x^2 + y^2 = r^2$$



The equation comes from $\sqrt{(x-h)^2 + (y-k)^2} = r$, which literally says (x, y) are points at distance $= r$ from the point (h, k).

Circles

The general form of a circle is the expanded version of the standard form:

$$x^2 + ax + y^2 + by + c = 0.$$

In order to find the center and the radius of the circle, simply complete the square with respect to (w.r.t.) x and y separately to convert the general form back to the standard form.

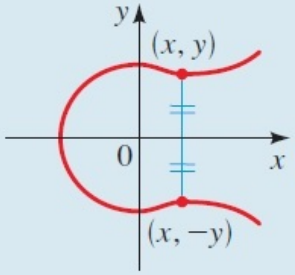
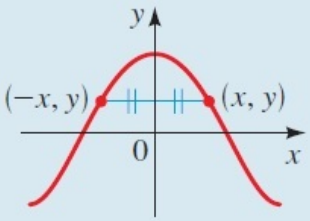
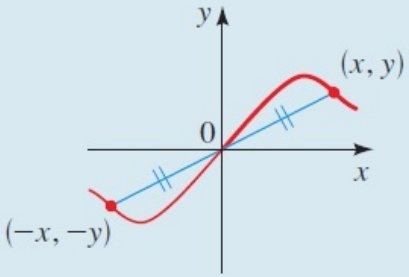
E.g.: Given $x^2 + y^2 + 2x - 6y + 7 = 0$, we group the x terms together and then the y terms together, after which we complete the square:

$$\begin{aligned}
& x^2 + y^2 + 2x - 6y + 7 = 0 \\
\Leftrightarrow \quad & (x^2 + 2x) + (y^2 - 6y) = -7 \\
\Leftrightarrow \quad & (x^2 + 2x + 1) + (y^2 - 6y + 9) = -7 + 1 + 9 \\
\Leftrightarrow \quad & (x + 1)^2 + (y - 3)^2 = 3 \\
\Leftrightarrow \quad & (x - (-1))^2 + (y - 3)^2 = (\sqrt{3})^2
\end{aligned}$$

Thus, it's the equation of the circle centered at $(-1, 3)$ with radius $= \sqrt{3}$.

Symmetry

TYPES OF SYMMETRY

Symmetry	Test	Graph	Property of Graph
With respect to the x-axis	Replace y by $-y$. The resulting equation is equivalent to the original one. E.g.: $y^2 = x^3 + 1$		Graph is unchanged when reflected about the x-axis.
With respect to the y-axis	Replace x by $-x$. The resulting equation is equivalent to the original one. E.g.: $y = 1 - x^2$		Graph is unchanged when reflected about the y-axis.
With respect to the origin	Replace x by $-x$ and y by $-y$. The resulting equation is equivalent to the original one. E.g.: $y = \dfrac{x}{1 + x^2}$		Graph is unchanged when rotated 180° about the origin.

Activity 3

(a) Find the x- and y-intercepts of the graphs of the following equations.

i) $y = x + 6$

ii) $y - 2xy + 4x = 1$

(b) Find the centers and radii of the following circles.

i) $(x + 1)^2 + y^2 = 9$

ii) $x^2 + y^2 + 4x - 6y + 12 = 0$

(c) Check whether the graphs of the following equations are symmetric w.r.t. the x-axis, y-axis or about the origin.

i) $x^2y^2 + xy = 1$

ii) $x = y^4 - y^2$

Activity 3: solution

(a) i) For x-intercept, let $y = 0$, so $0 = x + 6 \Leftrightarrow x = -6$. For y-intecept, let $x = 0$, so $y = 6$.

ii) For x-intercept, let $y = 0$, so $0 + 4x = 1 \Leftrightarrow x = \frac{1}{4}$. For y-intecept, let $x = 0$, so $y = 1$.

(b) i) $(x+1)^2 + y^2 = 9 \Leftrightarrow (x - (-1))^2 + (y - 0)^2 = 3^2$, so the center is $(-1, 0)$ and the radius is 3.

ii) $x^2 + y^2 + 4x - 6y + 12 = 0 \Leftrightarrow x^2 + 4x + 4 + y^2 - 6y + 9 = -12 + 4 + 9 \Leftrightarrow (x+2)^2 + (y-3)^2 = 1$, so the center is $(-2, 3)$ and the radius is 1.

Activity 3: solution

(c) i) For symmetry w.r.t. the x-axis, we replace y by $-y$ in the equation: $x^2(-y)^2 + x(-y) = 1 \Leftrightarrow x^2y^2 - xy = 1$, which is not the same as the original equation. Hence the graph is not symmetric w.r.t. the x-axis.

For symmetry w.r.t. the y-axis, we replace x by $-x$ in the equation: $(-x)^2y^2 + (-x)y = 1 \Leftrightarrow x^2y^2 - xy = 1$, which is not the same as the original equation. Hence the graph is not symmetric w.r.t. the y-axis.

For symmetry about the origin, we replace x and y by $-x$ and $-y$ respectively in the equation $(-x)^2(-y)^2 + (-x)(-y) = 1 \Leftrightarrow x^2y^2 + xy = 1$, which is the same as the original equation, hence the graph is symmetric about the origin.

Activity 3: solution

(c) ii) For symmetry w.r.t. the x-axis, we replace y by $-y$ in the equation: $x = (-y)^4 - (-y)^2 \Leftrightarrow x = y^4 - y^2$, which is the same as the original equation. Hence the graph is symmetric w.r.t. the x-axis.

For symmetry w.r.t. the y-axis, we replace x by $-x$ in the equation: $-x = y^4 - y^2$, which is not the same as the original equation. Hence the graph is not symmetric w.r.t. the y-axis.

For symmetry about the origin, we replace x and y by $-x$ and $-y$ respectively in the equation $-x = (-y)^4 - (-y)^2 \Leftrightarrow -x = y^4 - y^2$, which is not the same as the original equation, hence the graph is not symmetric about the origin.

You can use Geogebra to visualize them.

Outline

1. Inequalities
2. The Coordinate Plane, Graphs of Equations, Circles
3. Lines
4. Summary
5. Equations [Class Participation Assignment]

Lines and the slope

A straight line in a coordinate plane has a constant "steepness", i.e. how quickly it rises (or falls) as we move from left to right. We quantify this as follows:

Definition:

The ***slope*** m of a non-vertical line that passes through the points $A(x_1, y_1)$ and $B(x_2, y_2)$ is the number

$$m = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1}.$$

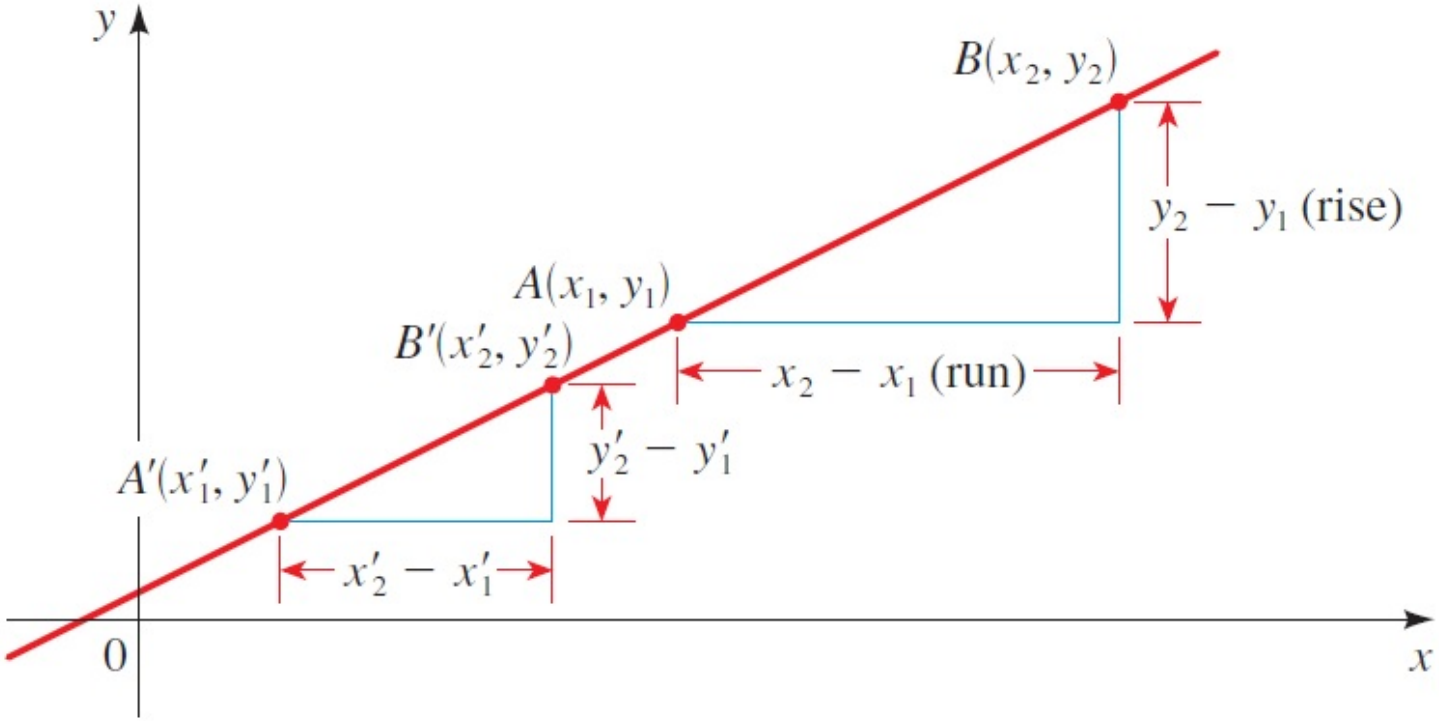
The slope of a vertical line is not defined.

Since $y_2 - y_1$ is the change in y value and $x_2 - x_1$ is the change in x value, the slope is also the rate of change of the y value of the line w.r.t. to the x value. Having a constant slope means lines have a constant rate of change of y w.r.t. x.

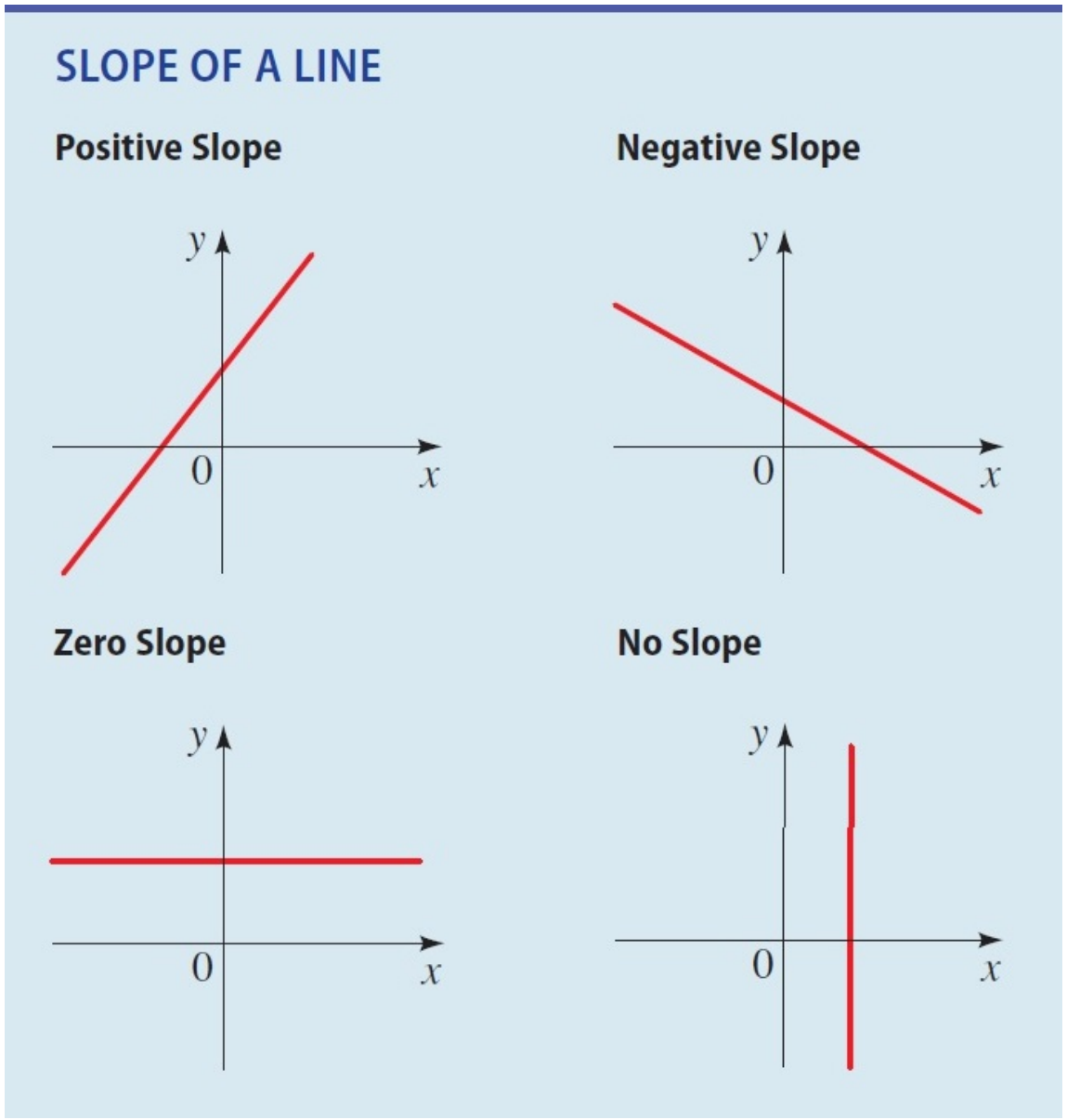
The slope of a line

The slope is independent of which two points are chosen on the line.

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{y_2' - y_1'}{x_2' - x_1'}$$



The slope of a line

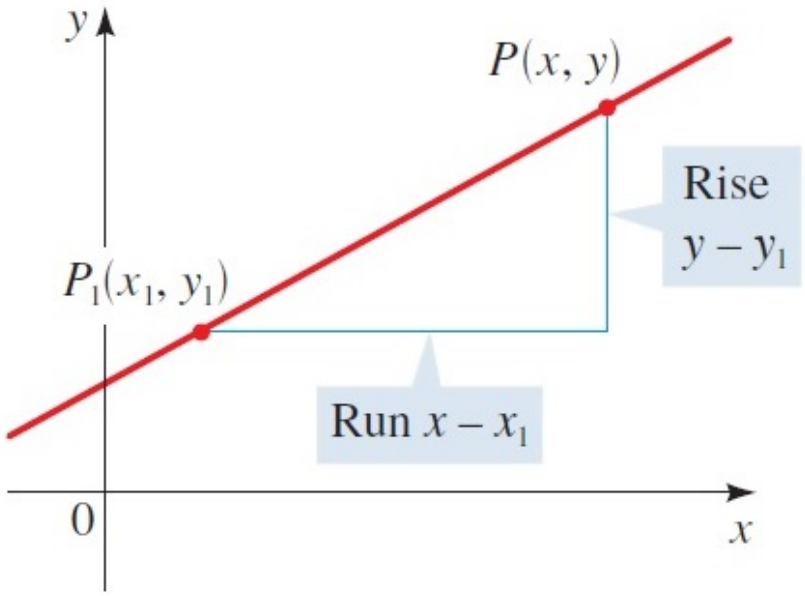


Point-Slope form of the equation of a line

Now let's find the equation of the line that passes through a given point $P_1(x_1, y_1)$ and has slope m. A point $P(x, y)$ with $x \neq x_1$ lies on this line iff. the slope of the line through $P1$ and P is equal to m, that is

$$\frac{y - y_1}{x - x_1} = m \quad \Leftrightarrow \quad y - y_1 = m(x - x_1).$$

The last equation is called the ***point-slope form*** of the equation of the line that passes through the point (x_1, y_1) and has slope m.



Point-Slope form of the equation of a line - example

E.g.: The point-slope form of the equation of the line that passes through $(-1, 3)$ and has slope $m = -2$ is

$$y - 3 = (-2)(x - (-1)) \;\Leftrightarrow\; y - 3 = -2(x + 1).$$

To find the equation of a line passing through two given points (x_1, y_1) and (x_2, y_2), we first use the two points to find the slope.

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

After that, we can apply the formula of the point-slope form of the equation of the line with any of the two points:

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) \quad \text{or} \quad y - y_2 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_2).$$

Point-Slope form of the equation of a line - example

E.g.: Given a line passing through $(1, 2)$ and $(-1, 3)$, then its slope is

$$m = \frac{3-2}{-1-1} = -\frac{1}{2}.$$

Thus, using the point $(1, 2)$, the point-slope form of the equation of the line is

$$y - 2 = -\frac{1}{2}(x - 1).$$

Note that we can also use the other point $(-1, 3)$ and get

$$y - 3 = -\frac{1}{2}(x - (-1)) \;\Leftrightarrow\; y - 3 = -\frac{1}{2}(x + 1).$$

Both equations are actually equivalent, as they both simplify to

$$y = -\frac{1}{2}x + \frac{5}{2}.$$

Slope-Intercept form of the equation of a line

If a line with slope m has y-intercept b, then the line passes through $(0, b)$. Applying the point-slope form formula with the point $(0, b)$ gives

$$y - b = m(x - 0) \qquad \Leftrightarrow \qquad y = mx + b.$$

The last equation is called the ***slope-intercept form*** of the equation of the line with slope m and y-intercept b.

⚠ For the special cases of vertical and horizontal lines, their equations are slightly different:

- An equation of the vertical line (x is constant) through (a, b) is $x = a$.
- An equation of the horizontal line (y is constant) through (a, b) is $y = b$.

General equation of a line

In order to express all kinds of straight lines (includng horizontal and vertical lines) as equations, we'll need the ***general equation of a line/linear equation***:

$$Ax + By + C = 0,$$

where A, B cannot be both zero.

- When $B \neq 0$, $Ax + By + C = 0 \;\Leftrightarrow\; y = -\frac{A}{B}x - \frac{C}{B}$. It's the line with slope $-\frac{A}{B}$ and y-intercept $-\frac{C}{B}$,
- When $B = 0$, then $A \neq 0$ and so $Ax + 0y + C = 0 \;\Leftrightarrow x = -\frac{C}{A}$. It's the vertical line passing through $\left(-\frac{C}{A}, 0\right)$.
- When $A = 0$, then $B \neq 0$ and so $0x + By + C = 0 \;\Leftrightarrow y = -\frac{C}{B}$. It's the horizontal line passing through $\left(0, -\frac{C}{B}\right)$.

Parallel and perpendicular lines

Definition:

Two lines are ***parallel*** iff. they are both vertical or they have the same slope.

Definition:

Two lines with slopes m_1 and m_2 are ***perpendicular***/***normal*** to each other iff. $m_1 \cdot m_2 = -1$, that is, their slopes are negative reciprocals

$$m_2 = -\frac{1}{m_1}.$$

Also, a horizontal line (slope 0) is perpendicular to a vertical line (no slope).

Please refer to page 112 of the textbook for the proof.

Activity 4

A line L_1 has the equation $3x - 2y + 4 = 0$.

(a) Find the slope, y-intercept and x-intercept of L_1.

(b) Another line L_2 passing through $(3, 2)$ is parallel to L_1. Find an equation of L_2.

(c) L_3 is perpendicular to L_1 and they both intersect at $x = 2$. Find an equation of L_3.

Activity 4: solution

(a) $3x - 2y + 4 = 0 \;\Leftrightarrow\; y = \frac{3}{2}x + 2$, so L_1 has slope $\frac{3}{2}$ and y-intercept 2. For the x-intercept, let $y = 0$, we get $3x - 0 + 4 = 0 \;\Leftrightarrow\; x = -\frac{4}{3}$.

(b) L_2 and L_1 have the same slope $\frac{3}{2}$. Thus, the point-slope form of the equation of L_2 is

$$y - 2 = \frac{3}{2}(x - 3).$$

Activity 4: solution

(c) The slope of L_3 is

$$m_2 = -\frac{1}{\frac{3}{2}} = -\frac{2}{3}.$$

Since the point of intersection also lies on L_1, we let $x = 2$ in the equation of L_1 and solve for y:

$$y = \frac{3}{2} \cdot 2 + 2 = 5.$$

So L_3 passes through $(2, 5)$ and has slope $-\frac{2}{3}$, which gives the equation

$$y - 5 = -\frac{2}{3}(x - 2).$$

Outline

1. Inequalities
2. The Coordinate Plane, Graphs of Equations, Circles
3. Lines
4. Summary
5. Equations [Class Participation Assignment]

Summary

- Inequalities
  - rules for inequalities.
  - solving linear inequalities.
  - solving nonlinear inequalities.
  - absolute value inequalities.
- The Coordinate Plane, Graphs of Equations, Circles
  - the coordinate plane.
  - the distance formula, the midpoint formula.
  - graphs of equations in two variables, intercepts, circles, symmetry.
- Lines
  - lines and the slope.
  - equation of a line: point-slope form, slope-intercept form, general linear equation
  - parallel and perpendicular lines.
- Equations [Class Participation Assignment]
  - definitions vs. equations.
  - Properties of equality.
  - solving quadratic equations, completing the square, the quadratic formula, the discriminant.
  - polynomial equations, rational equations.
  - equations involving radicals, absolute values.
  - rejecting extraneous solutions.
  - substitution method.

Resources

References:

- James Stewart. Precalculus Mathematics for Calculus (7th edition) sections 1.5, 1.8, 1.9 1.10. Sections 1.6, 1.7, 1.11, 1.12 are not covered in this course, but it's good to know.

Additional resources:

- A video on the formula of solving cubic polynomials.

Outline

1. Inequalities
2. The Coordinate Plane, Graphs of Equations, Circles
3. Lines
4. Summary
5. Equations [Class Participation Assignment]

Definitions

Notation:

The colon equal sign $:=$ is used to define new symbols. $A := B$ means A is defined to be B. We use this to introduce/define the new symbol A, whereas B is something that we already know.

E.g.:

- An apple $:=$
- $\mathbb{Q} :=$ the set of all rational numbers.

One cannot question the truth of a definition. It is defined as it is, although one can question the choice of the new symbol/word, but this is just matter of taste or historical background (why don't we call a Papple? Bapple? Capple, etc.).

Equations

Notation:

The equal sign $=$ is used to denote an equation, i.e. a mathematical statement where the left-hand side is equal to the right-hand side. When we write $a = b$, we claim that a is equal to b. The claim can be true or false.

E.g.:

- $1 + 1 = 2$ is true.
- $0 = 1$ is false.
- $4x + 7 = 19$ is true for only some x.
- $\frac{2}{4} = \frac{1}{2}$ is true.

As you can see from the last example, some equations involve unknowns, and by solving an equation, we mean to find all the values of the unknowns such that the equation is true (called the ***solutions*** or ***roots***). To do this, we try to find a simpler, equivalent equation (two equations are equivalent if one implies the other and vice-versa, denoted as $\Leftrightarrow$) and repeat this step until we can explicitly get the values of the unknown.

Properties of equality

The following are two key properties that allow us to get a simpler, equivalent equation.

PROPERTIES OF EQUALITY

Property	Description
1. $A = B \Leftrightarrow A + C = B + C$	Adding the same quantity to both sides of an equation gives an equivalent equation.
2. $A = B \Leftrightarrow CA = CB \quad (C \neq 0)$	Multiplying both sides of an equation by the same nonzero quantity gives an equivalent equation.

Property 1 explains why you "move" a $+7$ from one side of an equation to the other side of an equation and it becomes -7:

$$
\begin{aligned}
& & 4x + 7 &= 19 \\
&\Leftrightarrow & \underbrace{4x + 7}_{A} + \underbrace{(-7)}_{C} &= \underbrace{19}_{B} + \underbrace{(-7)}_{C} \\
&\Leftrightarrow & 4x &= 19 - 7.
\end{aligned}
$$

Properties of equality

Similarly, property 2 explains why you "move" a coefficient 4 from one side of an equation to the other side of an equation and it becomes $\frac{1}{4}$:

$$\begin{aligned}
\Leftrightarrow \quad & 4x = 12 \\
\Leftrightarrow \quad & \underbrace{\frac{1}{4}}_{C} \times \underbrace{4x}_{A} = \underbrace{\frac{1}{4}}_{C} \times \underbrace{12}_{B} \\
\Leftrightarrow \quad & x = \frac{1}{4} \times 12 = 3.
\end{aligned}$$

You can apply these properties to solve ***linear equations***, i.e. equations of the form $ax + b = cx + d$.

Solving quadratic equations

Definition:

A ***quadratic equation*** is an equation of the form $ax^2 + bx + c = 0$, where a, b and c are real numbers and $a \neq 0$.

To solve a quadratic equation, we first factorize it as $ax^2 + bx + c = (px + r)(qx + s) = 0$, and then use the following property:

ZERO-PRODUCT PROPERTY

$$AB = 0 \quad \text{if and only if} \quad A = 0 \quad \text{or} \quad B = 0$$

Thus,

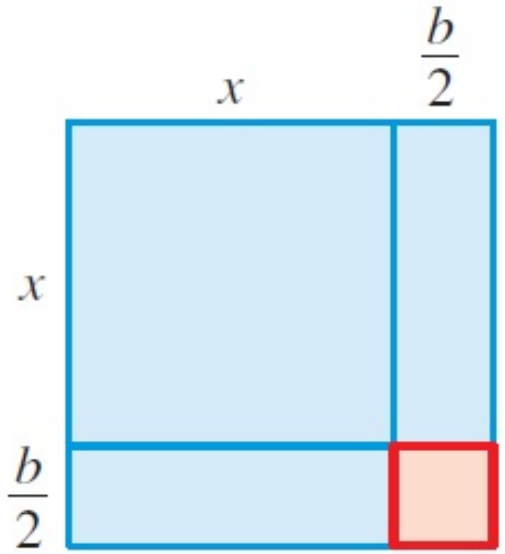
$$\begin{aligned}(px + r)(qx + s) = 0 \quad &\Leftrightarrow \quad px + r = 0 \quad \text{or} \quad qx + s = 0 \\ &\Leftrightarrow \quad x = -\frac{r}{p} \quad \text{or} \quad x = -\frac{s}{q}.\end{aligned}$$

However, factorizing a quadratic might be difficult, especially when the roots are not rational. A more general formula would be useful.

Completing the square

Since we know
$(A+B)^2 = A^2 + 2BA + B^2 = A^2 + (2B)A + \left(\frac{2B}{2}\right)^2$, to make $x^2 + bx$ a perfect square, we need to add $\left(\frac{b}{2}\right)^2$, the square of the half the coefficient of x. This gives the perfect square

$$x^2 + bx + \left(\frac{b}{2}\right)^2 = \left(x + \frac{b}{2}\right)^2.$$



The quadratic formula

Hence, given a general quadratic equation $ax^2 + bx + c = 0$, we first complete the square and then take the square root both sides to find x:

$$
\begin{aligned}
ax^2 + bx + c = 0 \quad &\Leftrightarrow \quad x^2 + \frac{b}{a}x + \frac{c}{a} = 0 \\
&\Leftrightarrow \quad x^2 + \frac{b}{a}x = -\frac{c}{a} \\
&\Leftrightarrow \quad x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2 \\
&\Leftrightarrow \quad \left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2} \\
&\Leftrightarrow \quad x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}} \\
&\Leftrightarrow \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.
\end{aligned}
$$

The discriminant

The radical term $\sqrt{b^2 - 4ac}$ in the formula tells us whether the roots are real numbers or not: as the squares of all real numbers are non-negative, so $\sqrt{b^2 - 4ac}$ is a real number iff. $b^2 - 4ac \geq 0$. Since we have two possible square roots $\pm\sqrt{b^2 - 4ac}$ whenever $b^2 - 4ac \neq 0$, the formula also tells us that the quadratic equation has two distinct roots iff. $b^2 - 4ac \neq 0$, otherwise when $b^2 - 4ac = 0$, there's only one single root $x = -\frac{b}{2a}$. All these motivate the following definition:

Definition:

The ***discriminant*** of a quadratic equation $ax^2 + bx + c = 0$ $(a \neq 0)$ is the number $D := b^2 - 4ac$.

- If $D > 0$, then the equation has two distinct real roots.
- If $D = 0$, then the equation has exactly one real root.
- If $D < 0$, then the equation has no real roots (in fact, it has two distinct non-real complex roots).

Solvability by radicals

Such general quadratic formula is handy because it's applicable to all quadratic equations. One might wonder are there such formulas for general higher degree polynomial equations? This is called the solvability by radicals problem for polynomials and it's been completely answered about 200 years ago, called the Abel–Ruffini theorem. Soon after Abel's publication of its proof, Évariste Galois introduced a theory, now called Galois theory that allows deciding, for any given polynomial equation, whether it is solvable in radicals.

Fun facts:

- There are formulas for the roots of a general cubic and a general quartic equations.
- There are no such formulas for the roots of a general polynomial equation of degree higher than 4. Mathematicians prove that no such formulas can exist.

Activity 5

Find all real roots of the following quadratic equations.

(a) $x^2 - 2x - 15 = 0$

(b) $2x^2 + 4x + 1 = 0$

(c) $9x^2 + 12x + 4 = 0$

(d) $x^2 = 3(x - 1)$

Activity 5: solution

(a)

$$x^2 - 2x - 15 = 0 \quad \Leftrightarrow \quad x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4 \cdot 1 \cdot (-15)}}{2 \cdot 1}$$

$$\Leftrightarrow \quad x = \frac{2 \pm \sqrt{64}}{2} = \frac{2 \pm 8}{2} = 5 \text{ or } -3.$$

(b)

$$2x^2 + 4x + 1 = 0 \quad \Leftrightarrow \quad x = \frac{-4 \pm \sqrt{4^2 - 4 \cdot 2 \cdot 1}}{2 \cdot 2}$$

$$\Leftrightarrow \quad x = \frac{-4 \pm \sqrt{8}}{4} = \frac{-4 \pm 2\sqrt{2}}{4}$$

$$= -1 + \frac{\sqrt{2}}{2} \text{ or } -1 - \frac{\sqrt{2}}{2}.$$

Activity 5: solution

(c)

$$9x^2 + 12x + 4 = 0 \quad \Leftrightarrow \quad x = \frac{-12 \pm \sqrt{12^2 - 4 \cdot 9 \cdot 4}}{2 \cdot 9}$$
$$\Leftrightarrow \quad x = \frac{-12 \pm 0}{18} = \frac{-2}{3}.$$

(d) $x^2 = 3(x-1) \quad \Leftrightarrow \quad x^2 - 3x + 3 = 0$. Since the discriminant $D = (-3)^2 - 4 \cdot 1 \cdot 3 = -3 < 0$, the equation has no real roots.

Polynomial equations

Definition:

A ***polynomial equation*** of degree n is an equation of the form $a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0 = 0$, where $a_n \neq 0$.

If we can completely factorize
$x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0 = (x - r_1)(x - r_2)\cdots(x - r_n) = 0$,
then we can apply the zero-product property on slide 57 again to conclude

$$\begin{aligned} (x - r_1)(x - r_2)\cdots(x - r_n) &= 0 \\ \Rightarrow \quad x - r_1 = 0 \quad \text{or} \quad x - r_2 = 0 \quad \ldots \text{ or} \quad x - r_n &= 0 \\ \Rightarrow \quad x = r_1 \quad \text{or} \quad x = r_2 \quad \ldots \text{ or} \quad x &= r_n. \end{aligned}$$

We'll return to this topic in a later week. The remainder and factor theorems can help us to factorize polynomials.

Rational equations

Definition:

A ***rational equation*** is an equation of the form $\frac{A}{B} = \frac{C}{D}$, where A, B, C, D are polynomials and $B, D \neq 0$.

The general rule of solving a rational equation $\frac{A}{B} = \frac{C}{D}$ is to cross-multiply and get the polynomial equation $AD = BC$. Then we proceed to solve the polynomial equation, but with the condition on the solutions such that $B, D \neq 0$. The reason behind this is because of the actual details of the cross-multiplication:

$$\frac{A}{B} = \frac{C}{D} \quad \Leftrightarrow \quad \frac{A}{B} - \frac{C}{D} = 0 \quad \Leftrightarrow \quad \frac{AD - BC}{BD} = 0$$

$$\Rightarrow \quad AB - BC = 0 \quad \text{and} \quad BD \neq 0.$$

Rational equations - example

E.g.: To solve the equation $\frac{3}{x} - \frac{2}{x-3} = \frac{-12}{x^2-9}$, we first simplify it into $\frac{3(x-3)-2x}{x(x-3)} = \frac{-12}{x^2-9}$ and we remark that $x(x-3) \neq 0$ and $x^2 - 9 \neq 0$, which is equivalent to $x \neq 0, \pm 3$.
Next, we cross-multiply and get

$$(x^2-9)(3(x-3)-2x) = -12x(x-3)$$
$$\Rightarrow \quad (x-3)(x+3)(3x-9-2x) + 12x(x-3) = 0$$

Note that we didn't hastily expand everything. We kept the factors instead and now we see that $(x-3)$ is a common factor.

$$\Rightarrow \quad (x-3)\left((x+3)(3x-9-2x)+12x\right) = 0$$
$$\Rightarrow \quad (x-3)(x^2+6x-27) = 0$$
$$\Rightarrow \quad (x-3)(x+9)(x-3) = 0.$$

Since previously we remarked that $x \neq 0, \pm 3$, so $x - 3 \neq 0$ and thus we must have $x + 9 = 0 \;\Leftrightarrow\; x = -9$.

Equations involving radicals

Suppose we want to solve an equation of the form $A = \sqrt{B}$, where A, B are polynomials, we first square both sides to get the polynomial equation $A^2 = B$ and proceed to solve it, but with the condition on the solutions such that $A \geq 0$ and $B \geq 0$. This is because $\sqrt{B}$ is only defined for $B \geq 0$ and $A = \sqrt{B} \geq 0$.

⚠ Recall that $\sqrt{B} = +\sqrt{B}$.

Equations involving radicals - example

E.g.: To solve $2x = 1 - \sqrt{2-x}$, we first move the radical to one side: $\sqrt{2-x} = 1 - 2x$. We then square both sides and bear in mind that $1 - 2x \geq 0$ and $2 - x \geq 0$:

$$
\begin{aligned}
& & (1-2x)^2 &= \left(\sqrt{2-x}\right)^2 = 2 - x \\
&\Rightarrow & 4x^2 - 4x + 1 - (2-x) &= 0 \\
&\Rightarrow & 4x^2 - 3x - 1 &= 0 \\
&\Rightarrow & (4x+1)(x-1) &= 0 \\
&\Rightarrow & 4x + 1 = 0 \quad &\text{or} \quad x - 1 = 0 \\
&\Rightarrow & x = -\frac{1}{4} \quad &\text{or} \quad x = 1.
\end{aligned}
$$

Upon checking the conditions $1 - 2x \geq 0$ and $2 - x \geq 0$, we reject $x = 1$ and conclude that $x = -\frac{1}{4}$ is the only solution.

Equations involving absolute values

Recall that if $|x| = 3$, then x could be -3 or 3. In other words, the solutions for $|x| = a > 0$ are $x = -a$ or $x = a$.

In view of this, if we want to solve an equation of the form $|A| = B$ where A, B are polynomials, it's equivalent to solve $A = B$ or $A = -B$, but with the condition on the solutions such that $B \geq 0$. Again, this is because $B = |A| \geq 0$.

E.g.: To solve $|2x - 5| = 3x$, it's equivalent to solve $2x - 5 = 3x$ or $2x - 5 = -3x$, while keeping in mind that $3x \geq 0$.

$$\begin{aligned} 2x - 5 &= 3x & \text{or} \quad & 2x - 5 = -3x \\ -5 &= x & \text{or} \quad & 5x = 5 \\ x &= -5 & \text{or} \quad & x = 1. \end{aligned}$$

But we need $3x \geq 0$, so we reject $x = -5$ and conclude that $x = 1$ is the only solution.

Rejecting extraneous solutions

By now you have seen that there are always extra conditions satisfied by the solutions for rational equations, equations involving radicals, equations involving absolute values and algebraic equations in general. If you forget to check these conditions, you might end up getting ***extraneous solutions***, i.e. solutions that do not satisfy the original equation. One easy way to check your solutions is to back substitute them into the original equation and verify that the equality holds.

E.g.: In the example of the previous page, suppose we didn't keep in mind of the condition $3x \geq 0$. When we get the two potential solutions $x = -5$ or $x = 1$, we substitute them into the original equation $|2x - 5| = 3x$. For $x = -5$, we get $|2(-5) - 5| = |-15| = 15 = 3(-5) = -15$, a contradiction. For $x = 1$, we get $|2 \cdot 1 - 5| = |-3| = 3 = 3 \cdot 1$, which is true.

⚠ It's possible that there are no extraneous solutions. It's also possible that all solutions are extraneous solutions, in which case there're no solutions to the original equation.

Substitution method

Lastly, there are algebraic equations which by first glance look difficult to be solved, but become easy and familiar after a tactful substitution. We illustrate this with two examples.

E.g.: To solve $x^4 - 8x^2 + 8 = 0$, we first substitute $y = x^2$ and get $y^2 - 8y + 8 = 0$. We then easily solve it by the quadratic formula:

$$y = \frac{-(-8) \pm \sqrt{(-8)^2 - 4 \cdot 1 \cdot 8}}{2} = \frac{8 \pm \sqrt{32}}{2} = 4 \pm 2\sqrt{2}.$$

Note that both $4 + 2\sqrt{2}$ and $4 - 2\sqrt{2}$ are positive. We back substitute $x^2 = y$:

$$\begin{aligned} & x^2 = 4 + 2\sqrt{2} && \text{or} && x^2 = 4 - 2\sqrt{2} \\ \Leftrightarrow \quad & x = \pm\sqrt{4 + 2\sqrt{2}} && \text{or} && x = \pm\sqrt{4 - 2\sqrt{2}} \end{aligned}$$

Substitution method - example

E.g.: To solve $x^{\frac{1}{3}} + x^{\frac{1}{6}} - 2 = 0$, we first substitute $y = x^{\frac{1}{6}}$ and get $y^2 + y - 2 = 0$. We then easily factorize it

$$y^2 + y - 2 = (y+2)(y-1) = 0.$$

Note that $y = -2 < 0$ and $y = 1 > 0$. Since $y = x^{\frac{1}{6}} \geq 0$, we reject $y = -2$ and conclude that $x^{\frac{1}{6}} = y = 1 \Rightarrow x = 1^6 = 1$.

Activity 6

(a) Find the roots/solutions of the following equations.

i) $\frac{1}{x} = \frac{4}{3x} + 1$

ii) $\frac{4}{x-1} + \frac{2}{x+1} = \frac{35}{x^2-1}$

iii) $\sqrt{2x+1} + 1 = x$

iv) $x^6 - 2x^3 - 3 = 0$

v) $x - 5\sqrt{x} + 6 = 0$

vi) $|3x+5| = x - 4$

(b) Express a in terms of b, given

$$\frac{a+1}{b} = \frac{a-1}{b} + \frac{b-1}{a}.$$

Activity 6: solution

(a) i)

$$\frac{1}{x} = \frac{4}{3x} + 1 \quad \Leftrightarrow \quad \frac{1}{x} - \frac{4}{3} \cdot \frac{1}{x} = 1$$

$$\Leftrightarrow \quad -\frac{1}{3} \cdot \frac{1}{x} = 1 \quad \Leftrightarrow \quad x = -\frac{1}{3}.$$

You can also solve it by substituting $y = \frac{1}{x}$.

Activity 6: solution

(a) ii)

$$
\begin{aligned}
& \frac{4}{x-1} + \frac{2}{x+1} = \frac{35}{x^2-1} \\
\Leftrightarrow \quad & \frac{4(x+1)+2(x-1)}{(x-1)(x+1)} = \frac{35}{(x-1)(x+1)} \\
\Leftrightarrow \quad & \frac{6x+2-35}{(x-1)(x+1)} = 0 \\
\Leftrightarrow \quad & \frac{6x-33}{(x-1)(x+1)} = 0 \\
\Leftrightarrow \quad & 6x-33 = 0 \quad \text{and} \quad x \neq \pm 1 \\
\Leftrightarrow \quad & x = \frac{33}{6} = \frac{11}{2}.
\end{aligned}
$$

Activity 6: solution

(a) iii)

$$\begin{aligned}
& \sqrt{2x+1}+1 = x && \Leftrightarrow \quad \sqrt{2x+1} = x-1 \\
\Rightarrow \quad & 2x+1 = (x-1)^2 && \Leftrightarrow \quad 2x+1 = x^2-2x+1 \\
\Leftrightarrow \quad & x^2-4x = 0 && \Leftrightarrow \quad x(x-4) = 0 \\
& && \Leftrightarrow \quad x = 0 \text{ or } x = 4.
\end{aligned}$$

When $x = 0$, $\sqrt{2x+1}+1 = \sqrt{0+1}+1 = 2 \neq x = 0$, so we reject $x = 0$.

When $x = 4$, $\sqrt{2x+1}+1 = \sqrt{8+1}+1 = 4 = x$, so $x = 4$ is the solution.

Activity 6: solution

(a) iv) We substitute $y = x^3$ and get

$$\begin{aligned} x^6 - 2x^3 - 3 = 0 \quad &\Leftrightarrow \quad y^2 - 2y - 3 = 0 \\ \Leftrightarrow \quad (y-3)(y+1) = 0 \quad &\Leftrightarrow \quad y = 3 \text{ or } y = -1 \\ &\Leftrightarrow \quad x^3 = 3 \text{ or } x^3 = -1 \\ &\Leftrightarrow \quad x = \sqrt[3]{3} \text{ or } x = -1. \end{aligned}$$

v) We substitute $y = \sqrt{x}$ and get

$$\begin{aligned} x - 5\sqrt{x} + 6 = 0 \quad &\Leftrightarrow \quad y^2 - 5y + 6 = 0 \\ \Leftrightarrow \quad (y-3)(y-2) = 0 \quad &\Leftrightarrow \quad y = 3 \text{ or } y = 2 \\ &\Leftrightarrow \quad \sqrt{x} = 3 \text{ or } \sqrt{x} = 2 \\ &\Leftrightarrow \quad x = 9 \text{ or } x = 4. \end{aligned}$$

Activity 6: solution

(a) vi) Solving $|3x+5| = x-4$ is equivalent to solve

$$\begin{aligned} 3x+5 &= x-4 && \text{or} && 3x+5 = -(x-4) \\ 2x &= -9 && \text{or} && 4x = -1 \\ x &= -\frac{9}{2} && \text{or} && x = -\frac{1}{4}. \end{aligned}$$

When $x = -\frac{9}{2}$,
$|3x+5| = \left|3\left(-\frac{9}{2}\right)+5\right| = \left|-\frac{17}{2}\right| = \frac{17}{2} \neq -\frac{9}{2}-4 = x-4$,
so we reject $x = -\frac{9}{2}$.
When $x = -\frac{1}{4}$,
$|3x+5| = \left|3\left(-\frac{1}{4}\right)+5\right| = \left|\frac{17}{4}\right| = \frac{17}{4} \neq -\frac{1}{4}-4 = x-4$,
so we reject $x = -\frac{1}{4}$.
We conclude that the equation has no solutions.
Geometrically, the graphs of the two functions $y = |3x+5|$ and $y = x-4$ do not intersect, visualize this with Geogebra.

Activity 6: solution

(b) We just treat a as the unknown x and apply the same method to solve for a:

$$
\begin{aligned}
& & \frac{a+1}{b} &= \frac{a-1}{b} + \frac{b-1}{a} \\
\Leftrightarrow & & \frac{a+1}{b} - \frac{a-1}{b} &= \frac{b-1}{a} \\
\Leftrightarrow & & \frac{a+1-(a-1)}{b} &= \frac{b-1}{a} \\
\Leftrightarrow & & \frac{2}{b} &= \frac{b-1}{a} \\
\Leftrightarrow & & a &= \frac{b(b-1)}{2}.
\end{aligned}
$$